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EFFICIENCY AND ACCURACY OF LINEARIZED POSTBUCKLING ANALYSES OF FRAMES BASED ON ELASTICA

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Abstract—The efficiency, accuracy and ease in use of linearized stability analyses for establishing the postbuckling response of rigid jointed frames is discussed using as a model a simple rectangular frame. This is accomplished by comparing a variety of results of the linearized stability analyses with those of the exact elastica analysis the governing equations of which appear for the first time in the technical literature; important conclusions on the accuracy and range of applicability of the foregoing linearized stability analyses are drawn. It is found that the completely linearized stability analysis, corresponding to a linear boundary-value problem, furnishes very reliable results which are quite near to those of the widely used non-linear kinematic stability analysis and much more accurate than those of the standard (incompressible) elastica; thus the former analysis, being the simplest possible postbuckling analysis, constitutes, for structural design purposes the most powerful method for determining the initial postbuckling response of frames.

1. INTRODUCTION

The problem of determining the postbuckling response of structural systems is of particular importance in modern engineering. A major difficulty for establishing such a response, even in the case of simple framed structures, is the intractability of the non-linear differential equations due to the non-linear bending moment–curvature relationship. In most cases exact solutions cannot be obtained or are very cumbersome and time consuming. Hence, often the only recourse is to resort to approximate stability analyses. An early outstanding contribution was the initial postbuckling analysis of Koiter (1945); this analysis constitutes a higher-order linearization of the governing equilibrium equations leading to accurate results for structural systems that lose their stability mainly through distinct bifurcational points. However, Koiter's analysis—besides its restrictions (Budiansky, 1974), in many applications—even in the case of a two-bar frame (Koiter, 1966), is not practical to use. Another contribution to this area is the initial postbuckling analysis presented by Roorda and Chilver (1970) based on a perturbation technique. Pertinent to this subject is the book by Britvec (1973) in which particular emphasis is given to non-linear stability analyses of rigid jointed frames. More simple and efficient stability analyses of frames have been presented (Kounadis *et al.*, 1977; Simitses and Kounadis, 1978), based on the so-called intermediate theory of deformation, valid for small strains and moderate rotations. However, even in the last systematic stability analysis, considerable difficulties arise for obtaining numerical solutions for frames having more than three bars. This is mainly due to an intrinsic reason of the theory of the intermediate class of deformation (Brush and Almroth, 1975). This theory, although based on linearized buckling equations, uses non-linear kinematic relations which imply considerable computational difficulties. To this end efficient computer algorithms for the post-buckling analysis of multistory and/or multibay frames were developed (Vlachinos *et al.*, 1986; Simitses *et al.*, 1986; Economou, 1984). Prior to the last works a simplified but more efficient non-linear stability analysis of frames, based on linear kinematic relations, was presented (Kounadis, 1985). In view of the simplified formulation of the governing equations of the last analysis its efficiency depends on the choice of the solution numerical scheme. This analysis is the simplest possible for establishing prebuckling and postbuckling equilibrium paths; despite this fact it leads to results which for rectangular frames differ (Kounadis, 1985), less than 6% from those of the foregoing non-linear kinematic stability analysis. But is the last, widely used analysis, so reliable as it is generally believed? Are the results more reliable when they are obtained by using non-linear instead of linear kinematic relations? To the knowledge of the author there is not yet any evidence with regard to the accuracy and range of applicability of both the foregoing stability analyses.

The objectives of this investigation, using as a model a simple rectangular frame for which available results exist, are given below.

(a) To elucidate the aforementioned questions using a thorough and comprehensive theoretical discussion supplemented by a large number of results covering a variety of practical applications. This is accomplished with the aid of the exact buckling equations of elastica analysis which under this form appear in the technical literature for the first time.

(b) To define which is the most efficient and practical postbuckling analysis for frames having a large number of bars; namely, a postbuckling analysis that can be applied by structural engineers whose mathematical training does not extend beyond the classical methods of analysis.

(c) To find out which parameters have an appreciable effect on the postbuckling response of this frame.

In the analysis presented herein the effect of compressibility of the bar axis is taken into account. Such an effect may be appreciable on the postbuckling response of structural systems losing their stability through a limit point. Reviewing the current state-of-the-art one should report the works by Huddleston (1967), Christodolou and Kounadis (1986), and Kounadis (1986) which neglect the aforementioned effect (standard elastica (Stoker (1968))). The first of these analyses gives some partial results based on a direct numerical solution of the governing differential equations of equilibrium, while the latter present an analytical solution through elliptic integrals. Finally, an elastica type analysis based also on a direct integration of the governing differential equations after some partial linearization, is presented by Quashu and Dadeppo (1983).

2. BASIC RELATIONS AND STATEMENT OF THE PROBLEM

The subsequent analysis is based on the theory of one-dimensional elastica theory, valid for large displacements but small strains compared to unity, according to the approximation

$$\frac{\mathrm{d}s-\mathrm{d}x}{\mathrm{d}x} = \sqrt{(1+2\varepsilon)}-1 \simeq \varepsilon \tag{1}$$

where ds is the final (after deformation) length of a line element with initial (before deformation) length dx. Moreover, it is assumed that plane sections normal to the undeformed bar axis remain plane and normal to the deformed axis. According to this theory the axial and lateral displacement components $U(x)$ and $W(x)$ at any point of the axis of a uniform Euler–Bernoulli straight bar subjected simultaneously to bending and compression are given by (Kounadis, 1986; Stoker, 1968)

$$U(x) = \int_0^x (1+\varepsilon)\cos\vartheta(x')\,\mathrm{d}x' + U(0) - x$$

$$W(x) = \int_0^x (1+\varepsilon)\sin\vartheta(x')\,\mathrm{d}x' + W(0) \tag{2}$$

where $\vartheta = \vartheta(x)$ and $\varepsilon = \varepsilon(x)$ are the rotation of the tangent and axial strain at an arbitrary point x of the bar axis, while x is referred to as the undeformed state.

Using the exact stability analysis based on the theory of elastica, one can subsequently discuss the accuracy and define the range of applicability of any other stability analysis. Since the interest of this work is focused on the non-linear simplified analysis (Kounadis, 1985) and the widely used non-linear kinematic stability analysis, the validity of the assumptions on which these analyses are based will be thoroughly discussed. The model which will

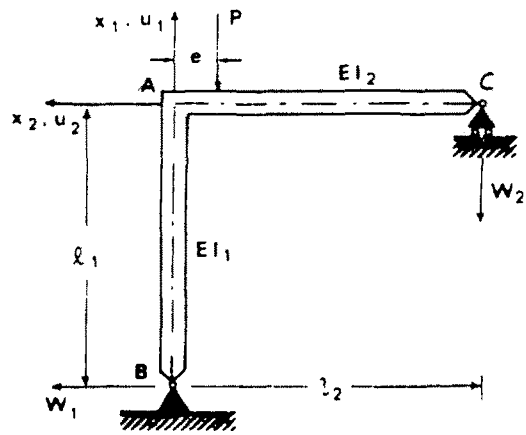

Fig. 1. Geometry and sign convention.

be used is the rectangular two-bar frame BAC shown in Fig. 1 for which numerical results are available. The chosen sign convention is also shown in this figure. The frame is subjected to an eccentrically applied force P at its joint A. The column is supported on an immovable hinge at point B, whereas the girder is free to move in the horizontal direction with the aid of a movable hinge (point C). Let l_i, A_i and I_i be the length, cross-sectional area and moment of inertia of the ith bar ($i = 1, 2$).

3. EXACT ELASTICA ANALYSIS

The deformation of the frame can be conveniently expressed in terms of the tangent of rotation $\vartheta_i(x)$ and axial strain $\varepsilon_i(x)$ at an arbitrary point x of the centre line of the ith bar ($i = 1, 2$).

Equating internal and external bending moments at arbitrary points of the vertical and horizontal member, respectively, one can write the following differential equations based on the exact bending moment–curvature relationship (Britvec, 1973; Kounadis, 1986):

$$
\begin{aligned}
-EI_1\vartheta_1' &= Sw_1 \\
-EI_2\vartheta_2' &= (P-S)[x_2+U_2(x_2)-U_2(0)]
\end{aligned}
\tag{3}
$$

where S is the vertical reaction at support B. Note that the eccentricity e is sufficiently small such that P can be replaced by *a* centrally applied to the column centre load P and a couple Pe.

Differentiation of eqns (3) and using relations (2) with the aid of expressions

$$
\begin{aligned}
\varepsilon_1 &= -\frac{S}{EA_1}\cos\vartheta_1 \\
\varepsilon_2 &= -\frac{P-S}{EA_2}\sin\vartheta_2
\end{aligned}
\tag{4}
$$

yields

$$
\begin{aligned}
EI_1\vartheta_1''+S\left(1-\frac{S}{EA_1}\cos\vartheta_1\right)\sin\vartheta_1 &= 0 \\
EI_2\vartheta_2''+(P-S)\left(1+\frac{P-S}{EA_2}\sin\vartheta_2\right)\cos\vartheta_2 &= 0.
\end{aligned}
\tag{5}
$$

These equations can also be derived by using an energy variational approach (Kounadis, 1986); they are exact or at least the most precise equilibrium equations of a frame that have appeared in the technical literature. Obviously, they are not subject to any restriction concerning the magnitude of rotations $\vartheta_i(x)$, while at the same time they include the effect of axial contraction (or extension) of the bar axis which is usually neglected (Britvec, 1973; Huddleston, 1967; Christodolou and Kounadis, 1986); the inextensional elastica is known as standard elastica (Stoker, 1968).

The boundary conditions associated with eqns (5) are

$$\begin{aligned} \vartheta_1(l_1)-\vartheta_2(l_2) &= 0 \\ EI_1\vartheta_1'(l_1)+EI_2\vartheta_2'(l_2)+P\epsilon &= 0 \\ \vartheta_1'(0) &= 0 \\ \vartheta_2'(0) &= 0 \end{aligned} \tag{6}$$

and

$$W_2(l_2) = -U_1(l_1). \tag{7}$$

The last one which is the kinematic continuity condition, by means of relations (2), becomes

$$\int_0^{l_2}(1+\varepsilon_2)\sin\vartheta_2(x_2)\,\mathrm{d}x_2+\int_0^{l_1}(1+\varepsilon_1)\cos\vartheta_1(x_1)\,\mathrm{d}x_1-l_1 = 0. \tag{8}$$

Introducing the dimensionless quantities

$$\begin{aligned} &x_i = \frac{x_i}{l_i}, \quad \Theta_i(x_i) = \vartheta_i[x_i(x_i)], \quad \lambda_i^2 = \frac{A_i l_1^2}{I_i} \quad (i = 1,2) \\ &\beta^2 = \frac{Pl_1^2}{EI_1}, \quad k^2 = \frac{Sl_1^2}{EI_1}, \quad \mu = \frac{I_2}{I_1}, \quad \rho = \frac{l_2}{l_1}, \quad e = \frac{\epsilon}{l_2} \end{aligned} \tag{9}$$

eqns (5), (6) and (8) become

$$\begin{aligned} \Theta_1''+k^2\left(1-\frac{k^2}{\lambda_1^2}\cos\Theta_1\right)\sin\Theta_1 &= 0 \\ \Theta_2''+\frac{\rho^2}{\mu}(\beta^2-k^2)\left(1+\frac{\rho^2}{\mu}\frac{(\beta^2-k^2)}{\lambda_2^2}\sin\Theta_2\right)\cos\Theta_2 &= 0 \end{aligned} \tag{10}$$

and

$$\begin{aligned} \Theta_1(1)-\Theta_2(1) &= 0 \\ \Theta_1'(1)+\frac{\mu}{\rho}\Theta_2'(1)+\rho\beta^2 e &= 0 \\ \Theta_1'(0) &= 0 \\ \Theta_2'(0) &= 0 \end{aligned}$$

$$\int_0^1\left(1-\frac{k^2}{\lambda_1^2}\cos\Theta_1\right)\cos\Theta_1\,\mathrm{d}x_1+\rho\int_0^1\left(1+\frac{\rho^2}{\mu}\frac{\beta^2-k^2}{\lambda_2^2}\sin\Theta_2\right)\sin\Theta_2\,\mathrm{d}x_2-1 = 0. \tag{11}$$

Without solving the above system of eqns (10) and (11) one can readily prove that the critical state of the perfect frame ($e = 0$) is associated—at least theoretically—with a limit point instability. In effect, the case of bifurcational instability must be excluded, because otherwise the equilibrium states on the primary path $\Theta_i(x_i) = 0$ (i.e. undeflected column and unstressed girder) should verify equilibrium equations, eqns (10) and (11), for all values of the load β^2. Clearly, this is not true unless $\lambda_1 \to \infty$ and $k^2 = \beta^2$. Consequently, for slenderness ratios $\lambda_1 \neq \infty$ the perfect frame loses its stability through a limit point.

An approximate variant of eqns (10) (Huddleston, 1967) with regard to the perfect frame ($e = 0$) obtained by neglecting the effect of contraction of the bar axis has been solved numerically by Runge–Kutta's scheme. A more reliable variant of eqns (10), where the term $\rho^2(\beta^2 - k^2) \sin \Theta_2/\mu\lambda_2^2$ is missing (being negligibly small compared to unity), has been successfully solved (Christodolou and Kounadis, 1986) through elliptic integrals. Moreover, a lot of numerical results covering a large range of values of the parameters ρ, μ, e and λ_1 were presented recently; they are based on both the simplified non-linear stability analysis and the widely used non-linear kinematic stability analysis (Kounadis, 1985, 1986).

In this investigation the "exact" stability analysis of the imperfect frame is accomplished through a direct numerical solution of eqns (10) using Runge–Kutta's numerical scheme. The resulting solutions covering the aforementioned large range of values of the above parameters are compared with those obtained by the foregoing two stability analyses under discussion.

In the sequel, the approximations and linearizations made in the exact equilibrium equations, eqns (10) and (11), which lead to the buckling equations of the last two stability analyses, are thoroughly discussed.

4. SIMPLIFIED NON-LINEAR ANALYSIS (KOUNADIS, 1985)

One can linearize eqns (10) and (11) by adopting the approximations

$$\begin{aligned} \sin \Theta &\simeq \Theta \\ \cos \Theta &\simeq 1 \end{aligned} \tag{12}$$

giving accurate results up to four significant figures for angles of rotation less than 0.017 rad ($\sim 1°$); namely, such a linearization is allowed inasmuch as the frame joint rotations at the vicinity of the critical state are smaller than 0.017 rad. Using linearizations (12) and the compatible approximation

$$\frac{\rho^2}{\mu\lambda^2}(\beta^2 - k^2) \sin \Theta_2 \simeq 0 \tag{13}$$

one can obtain the following equations corresponding to the linear boundary-value problem

$$\begin{aligned} \Theta_1'' + \overset{*}{k}{}^2\Theta_1 &= 0 \\ \Theta_2'' + \frac{\rho^2}{\mu}(\beta^2 - k^2) &= 0 \end{aligned} \tag{14}$$

$$\begin{aligned} \Theta_1(1) - \Theta_2(1) &= 0 \\ \Theta_1'(1) + \frac{\mu}{\rho}\Theta_2'(1) + \beta^2\rho e &= 0 \\ \Theta_1'(0) &= 0 \\ \Theta_2'(0) &= 0 \end{aligned}$$

$$\int_0^1 \left(1 - \frac{k^2}{\lambda_1^2}\right) \mathrm{d}x_1 + \rho \int_0^1 \Theta_2(x_2)\, \mathrm{d}x_2 - 1 = 0 \tag{15}$$

where

$$\overset{*}{k}^2 = k^2(1-k^2/\lambda_1^2). \tag{16}$$

Due to approximations (12) one should take for consistency $\overset{*}{k}^2 = k^2$, since for the low value of $\lambda_1 = 40$ one has $\max(k^2/\lambda_1^2) < 0.0009$. Integrating eqns (14) and using the first four boundary conditions (15) result in

$$\Theta_1(x_1) = \frac{\rho[k^2+\beta^2(e-1)]}{k \sin k} \cos kx_1$$

$$\Theta_2(x_2) = \frac{\rho^2(k^2-\beta^2)}{2\mu} x_2^2 + \frac{\rho[k^2+\beta^2(e-1)]}{k \tan k} + \frac{\rho^2(\beta^2-k^2)}{2\mu}. \tag{17}$$

Introducing expressions (17) into boundary condition $(15)_5$, yields the following non-linear equilibrium equation:

$$\frac{k^2+\beta^2(e-1)}{k \tan k} + \frac{\rho(\beta^2-k^2)}{3\mu} - \frac{k^2}{\lambda_1^2\rho^2} = 0 \tag{18}$$

which coincides with eqn (18a) in the analysis by Kounadis (1985). Equation (18) includes the effect of axial contraction, whereas it neglects the effect of axial displacement due to the bending of the column centre line.

The limit of stability is established by means of the condition

$$\frac{d\beta}{dk} = 0. \tag{19}$$

The critical (limit point) load corresponding to a maximum of the β–k curve is the smallest load which satisfies eqns (18) and (19) together with the inequality

$$d^2\beta/dk^2 < 0. \tag{20}$$

The last condition should always be checked in order to exclude the value of the load corresponding to the minimum of the physically unacceptable complementary path.

The analytical expression of eqn (19) is given by (Kounadis, 1985)

$$-\frac{1}{2}\left(\frac{k^2+\beta^2(e-1)}{k^2 \sin k}\right)(2k+\sin 2k)+2\cos k-2\left(\frac{1}{\lambda_1^2\rho^2}+\frac{\rho}{3\mu}\right)k \sin k = 0. \tag{21}$$

In view of the approximations made above the last approach is a completely linearized stability analysis.

One can also express the above equations in terms of $w_i(x_i) = W_i(x_i)/l_i$ and $u_i(x_i) = U_i(x_i)/l_i$. Thus, eqns (2) for the ith bar are written in dimensionless form as

$$u_i(x_i) = \int_0^{x_i} (1+\varepsilon_i) \cos \Theta_i(x_i')\, dx_i' + u_i(0) - x_i$$

$$w_i(x_i) = \int_0^{x_i} (1+\varepsilon_i) \sin \Theta_i(x_i')\, dx_i' + w_i(0). \tag{22}$$

Linearization of these equations after taking into account relations (4) and the boundary conditions $u_1(0) = w_i(0) = 0 (i = 1, 2)$ implies

$$u_1(x_1) = \int_0^{x_1}\left(1-\frac{k^2}{\lambda_1^2}\right) dx_1' - x_1 = -\frac{k^2}{\lambda_1^2}x_1$$

$$u_2(x_2) = \int_0^{x_2}\left(1+\frac{\rho^2}{\mu}\frac{(\beta^2-k^2)}{\lambda_2^2}\Theta_2(x_2')\right) dx_2' + u_2(0) - x_2 \simeq u_2(0) \tag{23}$$

and

$$w_1(x_1) = \int_0^{x_1}\left(1-\frac{k^2}{\lambda_1^2}\right)\Theta_1(x_1')\, dx_1'$$

$$w_2(x_2) = \int_0^{x_2}\Theta_2(x_2')\, dx_2'$$

or due to eqns (17)

$$w_1(x_1) = \frac{\rho[k^2+\beta^2(e-1)]}{k^2 \sin k}\sin kx_1$$

$$w_2(x_2) = \frac{\rho^2(k^2-\beta^2)}{6\mu}x_2^3 + \left\{\frac{\rho[k^2+\beta^2(e-1)]}{k \tan k} - \frac{\rho^2(k^2-\beta^2)}{2\mu}\right\}x_2. \tag{24}$$

Obviously, differentiation of these equations leads to eqns (17).

The buckling mode eqns (23) and (24) of the linearized stability analysis coincide with those of non-linear kinematic stability analysis given by eqns (6) (Kounadis, 1985), with the unique exception that in eqns (23) the integral terms

$$0.5\int_0^{x_i} w_i'^2\, dx_i \quad (i = 1,2)$$

are missing. Such an omission which simplifies substantially the stability analysis is apparently consistent with the linearization accomplished above; whether or not it is worth retaining these terms together with the term $-k^2x_1/\lambda_1^2$ is a question which will be discussed below.

4.1. *Discussion of eqns* (22)

Differentiation of eqns (22) yields

$$u_i' = (1+\varepsilon_i)\cos\Theta_i - 1 = \cos\Theta_i - 1 + \varepsilon_i\cos\Theta_i$$
$$w_i' = (1+\varepsilon_i)\sin\Theta_i = \sin\Theta_i + \varepsilon_i\sin\Theta_i. \tag{25}$$

If the bar axis is considered as incompressible (which implies that ε_i should be neglected compared to unity) one can obtain

$$u_i' = \cos\Theta_i - 1 = \sqrt{(1-\sin^2\Theta_i)} - 1$$
$$w_i' = \sin\Theta_i. \tag{26}$$

From the last relations, it follows that

$$u_i' = \sqrt{(1-w_i'^2)} - 1 \simeq -\tfrac{1}{2}w_i'^2$$

and hence

$$\varepsilon_i = u_i' + \tfrac{1}{2}w_i'^2 = 0. \tag{27}$$

Note that without making use of the above approximation $\sqrt{(1-w_i'^2)} \cong 1-0.5w_i'^2$, one could also obtain from relation (26)

$$w_i'^2 = 4\sin^2\frac{\Theta_i}{2}\left(1-\sin^2\frac{\Theta_i}{2}\right) = -2u_i'(1+\tfrac{1}{2}u_i').$$

Hence

$$\varepsilon_i = u_i' + \tfrac{1}{2}(u_i'^2 + w_i'^2) = 0. \tag{28}$$

Thus, for the frame under consideration one can write

$$u_1(x_1) = -\frac{1}{2}\int_0^{x_1} w_1'^2 \, dx_1'$$

$$u_2(x_2) = C - \frac{1}{2}\int_0^{x_2} w_2'^2 \, dx_2' \tag{29}$$

where the constant $C = u_2(0)$—representing the horizontal displacement of the movable support—does not have any influence on the non-linear response of the frame.

Comparing the expressions of u_1 and u_2 of eqns (23) and (29) one can observe that the last ones (which represent the axial displacement due to bending of the column axis) are nonlinear. Clearly, they are time consuming and furnish the same computational complexities as those of the non-linear kinematic stability analysis due to the presence of the non-linear term $0.5w_i'^2$ in the expression of ε_i. The non-linear equilibrium equation based on eqn $(29)_1$, by virtue of condition (8), becomes

$$u_1(1) + \rho w_2(1) = 0$$

or

$$\rho w_2(1) - \frac{1}{2}\int_0^1 w_1'^2 \, dx_1 = 0. \tag{30}$$

This kinematic continuity condition with the aid of eqns (24) yields the equilibrium eqn (18b), given in Kounadis (1985), which is based on the incompressibility assumption of bar axes. Application of condition (19) to the last eqn (18b) leads to eqn (28) given by Kounadis (1985) which defines the limit of stability of frames having incompressible bars.

For small angles of rotation such that approximations (12) can be adopted, one can obtain a more accurate expression of u_i' than that given in relation (27), if the effect of compressibility of bar axes is taken into account. In this case relation $(25)_1$ becomes

$$\varepsilon_i = u_i'. \tag{31}$$

Clearly, the term $0.5w_i'^2$ is missing from the expression of axial strain $\varepsilon_i(=u_i'+0.5w_i'^2)$.

A more accurate kinematic relation than the last one can be obtained from relations (25) by adopting instead of approximations (12) the better ones

$$\sin\Theta_i \simeq \Theta_i$$

$$\cos\Theta_i = \sqrt{(1-\Theta_i^2)} \simeq 1 - \frac{\Theta_i^2}{2} \tag{32}$$

valid within four significant figures of accuracy for angles of rotation less than 0.052 rad

($\sim 3°$); namely, three times larger than those of approximations (12). Relations (25) using approximations (32) become

$$u_i' = -\frac{\Theta_i}{2} + \varepsilon_i\left(1 - \frac{\Theta_i^2}{2}\right) \simeq -\frac{\Theta_i^2}{2} + \varepsilon_i$$

$$w_i' = \Theta_i + \varepsilon_i\Theta_i \simeq \Theta_i \tag{33}$$

and hence

$$\varepsilon_i = u_i' + \tfrac{1}{2}w_i'^2. \tag{34}$$

Thus, kinematic relation (31) of the simplified non-linear stability analysis is more accurate than that given in relation (27) (or eqn (28)) which is based on the incompressibility assumption of bar axes. For rotations larger than 0.017 rad linear kinematic relation (31) must be replaced by the more accurate non-linear kinematic relation (34).

5. NON-LINEAR KINEMATIC STABILITY ANALYSIS

This analysis corresponds to an intermediate class of deformations (i.e. small strains with moderately large rotations such as $w_i'^2 \ll 1$) associated with non-linear kinematic relations ($\varepsilon_i = u_i' + 0.5w_i'^2$) combined with a linear moment–curvature relationship ($M_i = -EI_i w_i''$). The accuracy of this analysis as well as of the simplified stability analysis (eqns (23) and (24)) and the approximate analysis based on axially incompressible bars (eqns (24) and (29)) will be discussed below with the aid of the exact equations of the elastic analysis.

The differential equations of the non-linear kinematic stability analysis for the axial and transverse displacement are (Kounadis *et al.*, 1977; Simitses and Kounadis, 1978)

$$\lambda_i^2(u_i' + \tfrac{1}{2}w_i'^2)' = 0$$

and $(i = 1,2)$

$$w_i'''' - \lambda_i^2[(u_i' + \tfrac{1}{2}w_i'^2)w_i']' = 0 \tag{35}$$

or due to relation $(35)_1$

$$w_i'''' - \lambda_i^2(u_i' + \tfrac{1}{2}w_i'^2)w_i'' = 0 \quad (i = 1,2). \tag{36}$$

Equations (35) imply

$$\varepsilon_i = 0 \quad (i = 1,2). \tag{37}$$

Equations (37) are in disagreement—at least theoretically—with the exact eqns (4) which in dimensionless form are written as

$$\varepsilon_1 = u_1' + \tfrac{1}{2}w_1'^2 = -\frac{k^2}{\lambda_i^2}\cos\Theta_1$$

$$\varepsilon_2 = u_2' + \tfrac{1}{2}w_2'^2 = \frac{\rho^2(\beta^2 - k^2)}{\mu\lambda_2^2}\sin\Theta_2. \tag{38}$$

From these equations it is deduced that the axial forces $EA_1\varepsilon_1$ and $EA_2\varepsilon_2$ vary—at least theoretically—along the length of both bars; however, in view of approximations (32) and (33), on which the derivation of the non-linear kinematic relation (34) was based, eqns (38) are simplified as

$$\varepsilon_1 = u_1' + \tfrac{1}{2}w_1'^2 = -\frac{k^2}{\lambda_1^2}\left(1 - \frac{\Theta_1^2}{2}\right) \simeq -\frac{k^2}{\lambda_1^2}$$

$$\varepsilon_2 = u_2' + \tfrac{1}{2}w_2'^2 = \frac{\rho^2(\beta^2 - k^2)}{\mu\lambda_2^2}\Theta_2 \simeq 0. \tag{39}$$

Differentiation of relations (39) lead to eqns (37) or eqn $(35)_1$; the latter is therefore consistent with approximations (32) which lead to the non-linear kinematic relations (34).

Relations (39) due to the geometric boundary condition

$$u_1(0) = 0 \tag{40}$$

yield

$$u_1(x_1) = -\frac{k^2}{\lambda_1^2}x_1 - \frac{1}{2}\int_0^{x_1} w_1'^2\,dx_1'$$

$$u_2(x_2) = C - \frac{1}{2}\int_0^{x_2} w_2'^2\,dx_2' \tag{41}$$

which are more accurate than relations (23), while the latter are more accurate than relations (29) based on the incompressibility assumption of bar axes.

Equations (36) by virtue of relations (39) become

$$w_1'''' + k^2 w_1'' = 0$$
$$w_2'''' = 0. \tag{42}$$

Equation (42) can also be derived from the exact buckling eqns (10) using approximations (32). This leads to

$$\Theta_1'' + k^2\left(1 - \frac{k^2}{\lambda_1^2}\right)\Theta_1 = 0$$

$$\Theta_2'' + \frac{\rho^2}{\mu}(\beta^2 - k^2) = 0. \tag{43}$$

Differentiating eqns (43) and using relation $(33)_2$, one obtains eqns (42). The boundary conditions associated with eqns (42) are

$$w_1(1) = \rho u_2(1)$$
$$w_2(1) = -\frac{1}{\rho}u_1(1)$$
$$w_1'(1) = w_2'(1)$$
$$w_1(0) = w_2(0) = 0$$
$$w_1''(0) = w_2''(0) = 0$$
$$w_1''(1) + \frac{\mu}{\rho}w_2''(1) + \beta^2\rho e = 0$$
$$w_1'''(1) + k^2 w_1'(1) = 0$$
$$(\beta^2 - k^2)\rho^2 + \mu w_2'''(1) = 0. \tag{44}$$

All these conditions are linear with respect to w_i, u_i and their derivatives; they can be derived from the corresponding exact equations, eqns (10) and (11), after linearization with the aid of approximations (32) and (33).

Integration of eqns (42) and using the last eight of conditions (44) give eqns (24) which are those of the simplified stability analysis. Introducing eqns (24) into condition $(44)_2$ and using eqn $(41)_1$, one obtains the equilibrium equation of the non-linear kinematic stability analysis (Kounadis, 1985)

$$\frac{\rho(\beta^2-k^2)}{3\mu}+\frac{k^2+\beta^2(e-1)}{k\tan k}-\frac{k^2}{\rho^2\lambda_1^2}-\left[\frac{k^2+\beta^2(e-1)}{2k\sin k}\right]\left(1+\frac{\sin 2k}{2k}\right)=0. \tag{45}$$

The limit of stability corresponding to eqn (45) can be established by means of (Simitses and Kounadis, 1978; Kounadis, 1985)

$$\left(\frac{k^2+\beta^2(e-1)}{k^2\sin k}\right)^2\left(\frac{3k\sin k}{4}+\frac{3\sin k\sin 2k}{8}+k^2\frac{\cos k}{2}\right)$$
$$-\left(\frac{k^2+\beta^2(e-1)}{k^2\sin k}\right)(2k+\sin k)+2\cos k-\left(\frac{1}{\rho^2\lambda_1^2}+\frac{\rho}{3\mu}\right)2k\sin k=0. \tag{46}$$

It is worth observing that eqn (45) can be derived directly from the exact eqns (10) and (11) by adopting instead of approximations (12) the more accurate ones given in relation (32). Use of the latter leads to eqns (14) and (15) of the simplified non-linear stability analysis with the only exception being condition $(15)_5$ which becomes

$$\int_0^1\left(1-\frac{k^2}{\lambda_1^2}-\frac{1}{2}\Theta_1^2\right)\mathrm{d}x_1+\rho\int_0^1\Theta_2\,\mathrm{d}x_2-1=0$$

or

$$-\frac{k^2}{\lambda_1^2}-\frac{1}{2}\int_0^1\Theta_1^2\,\mathrm{d}x_1+\rho\int_0^1\Theta_2\,\mathrm{d}x_2=0. \tag{47}$$

Equation (47) by means of relation (17) leads to eqn (45).

Neglecting the first integral in relation (47) one can obtain condition $(15)_5$ which leads to the equilibrium equation, eqn (18), of the simplified non-linear stability analysis; therefore the latter is less accurate than eqn (45) valid for larger rotations than those of the simplified non-linear stability analysis. However, comparing the magnitudes of the two integrals in relation (47) it is clear that the first one could be omitted for conventional frames in which the length ratio ρ is not much smaller than 1. This is confirmed by a large variety of existing results (Economou and Kounadis, 1987; Kounadis, 1985).

From the foregoing theoretical discussion, it is deduced that the simplified non-linear stability analysis is less accurate—at least theoretically—than the time-consuming non-linear kinematic stability analysis.

If now the first term in relation (47) is omitted the resulting equation using relations (17) leads to equilibrium eqn (18b), in Kounadis (1985), based on the incompressibility assumption of bar axes. Hence, this analysis should be compared with the standard elastica (Stoker, 1968) in which this effect is ignored. The inaccuracy of this equation—being associated with bifurcational instability—increases as the slenderness ratio λ_1 decreases.

As is known the critical bifurcational loads are greater than the corresponding (critical) limit point loads of both the simplified and the non-linear kinematic stability analyses. The bifurcational stability analysis is less accurate than the simplified non-linear stability analysis because the latter is based on kinematic relation (31) which—as was shown—is more accurate than relation (27) (or relation (28)). On the other hand, it was shown that the exact equation, eqns (10) and (11), are associated—at least theoretically—with a limit point instability. Thus, the equilibrium equation of the simplified non-linear stability analysis resulting from eqn (47)—being associated with a limit point instability—should be more accurate than the corresponding equation of the bifurcational analysis resulting also from eqn (47).

Before closing this section it is worth mentioning that the majority of framed structures lose their elastic stability through a limit point; however, regarding usual cases of multistory building frames a bifurcational or a second-order analysis (Iffland, 1978; Wang, 1986a) which neglects the bar axial deformation could be employed. Recently an attempt (Wang, 1986b), has been made to decrease the error due to the ignorance of this effect by using an approximately equivalent reduction of member stiffness. In this work an interesting computer algorithm for the second-order analysis of multistory building frames has also been presented. Finally, it is worthwhile to note that second-order analyses cannot predict the actual load-carrying capacity of frameworks exhibiting postbuckling strength, while their application to imperfection sensitive frames (associated with the catastrophic failure of snapping) should be avoided.

6. NUMERICAL RESULTS

In this section numerical results obtained by the non-linear kinematic stability analysis (eqns (45) and (46)) of the completely linearized stability analysis (eqns (18) and (21)) and the stability analysis that neglects the effect of compressibility of bar axes (eqns (18b) and (28) in Kounadis (1985)), are compared with those of the elastica stability analysis (eqns (10) and (11)). These results correspond to a large variety of values of the parameters μ $(=0.25, 1, 4)$, ρ $(=0.25, 1, 4)$, $\lambda_1 = \lambda_2 = \lambda$ $(=40, 80, 120)$ and e $(=0, -0.0025)$ including also extreme cases. Thus, for $\lambda_1 = \lambda_2$ implying $A_2 = (\mu/\rho^2)/A_1$ the (largest) value of $\mu = 4$ combined with the (smallest) value of $\rho = 0.25$ leads to $A_2 = 64A_1$; obviously, such values of parameters do not correspond to any practically important frame having members with solid cross-sections but rather to the extreme case of a frame with built-up members.

The numerical solution of the exact elastica stability problem is established by a direct solution of eqns (10) and (11) using Runge–Kutta's scheme, whereas the solutions of the other three approximate stability analyses are obtained by using the Newton–Raphson technique. Once the solution for different levels of the loading β^2 and various values of the foregoing parameters is obtained by each one of the aforementioned analyses, the corresponding equilibrium states are established as functions of β^2 vs the horizontal joint displacement $w_1(1)$. The evaluation of the critical loads and critical displacements is accomplished by each one of the three linearized stability analyses through the solution of a system of two non-linear equations. On the contrary, the evaluation of these critical quantities by the elastica stability analysis is performed by step increasing the loading β^2 as a maximum in the curve β^2 vs $w_1(1)$, where $w_1(1)$ is given in relation (24). The variety of results presented in Tables 1–3 is obtained by the elastica analysis, the bifurcational stability analysis (Kounadis, 1985) based on the incompressibility assumption of bar axes, the completely linearized stability analysis and the non-linear kinematic stability analysis (Economou, 1984; Kounadis *et al.*, 1977; Kounadis, 1985).

From Table 1, evaluated for $e = 0$ and $\mu = 1$, one can see the effect of slenderness ratio and length ratio on the critical load β_{cr}^2 and the critical displacement $w_1(1)$, determined by using the foregoing four different analyses. Clearly, among the three linearized analyses that which is based on the incompressibility assumption of bar axes is less accurate than the other two analyses; at the same time this stability analysis is time consuming and cumbersome like the non-linear kinematic stability analysis due to the non-linear expression of axial displacements given by relations (29). Moreover, it is worth observing that the completely linearized analysis (eqns (18) and (21)) gives slightly less accurate results than the widely used non-linear kinematic stability analysis (eqns (45) and (46)), valid for moderate rotations which is associated with the computational disadvantages outlined above. The major discrepancies of both the foregoing stability analyses, appear for low values of length ratio ρ and slenderness ratio λ. Thus, for $\rho = 0.25$ and $\lambda_1 = \lambda_2 = 40$ the completely linearized analysis gives an error in the critical loads of 4.6%, whereas the

Table 1. Critical loads and displacements for $e = 0$, $\mu = I_2/I_1 = 1$, and various values of ρ and λ

		Exact elastica analysis, eqns (10) and (11)		Bifurcational stability analysis, eqns (18b) and (28) of Kounadis (1985)		Simplified non-linear stability analysis, eqns (18) and (21)		Non-linear kinematic stability analysis, eqns (45) and (46)	
λ_1	$\rho = l_1/l_2$	β_{cr}^2	$w_1(1) \times 10^{-2}$	β_{cr}^2	$w_1(1)$	β_{cr}^2	$w_1(1) \times 10^{-2}$	β_{cr}^2	$w_1(1) \times 10^{-2}$
	0.25	1.51788	2.5	2.10395	0	1.58771	2.3	1.51500	2.5
40	1	1.32617	2.3	1.42196	0	1.34013	2.3	1.32489	2.3
	4	0.58828	1.5	0.59500	0	0.58928	1.7	0.58804	1.6
	0.25	1.78508	1.3	The above results which do not change for $\lambda = 80$ and 120 coincide practically with those of standard elastica (Christodolou and Kounadis, 1986)		1.82700	1.2	1.78447	1.3
80	1	1.37283	1.2			1.38042	1.1	1.37249	1.2
	4	0.59156	0.7			0.59213	0.9	0.59151	0.8
	0.25	1.88519	0.9			1.91490	0.8	1.88490	0.9
120	1	1.38892	0.8			1.39413	0.8	1.38877	0.8
	4	0.59269	0.5			0.59309	0.6	0.59267	0.5

Table 2. Critical loads and displacements for $e = -0.0025$, $\mu = I_2/I_1 = 1$, and various values of ρ and λ

		Exact elastica analysis, eqns (10) and (11)		Bifurcational stability analysis, eqns (18b) and (28) of Kounadis (1985)		Simplified non-linear stability analysis, eqns (18) and (21)		Non-linear kinematic stability analysis, eqns (45) and (46)	
λ	$\rho = l_2/l_1$	β_{cr}^2	$w_1(1) \times 10^{-2}$	β_{cr}^2	$w_1(1) \times 10^{-2}$	β_{cr}^2	$w_1(1) \times 10^{-2}$	β_{cr}^2	$w_1(1) \times 10^{-2}$
	0.25	1.50954	2.5	2.00187	0.4	1.57970	2.3	1.50766	2.5
40	1	1.27626	3.5	1.30880	2.6	1.29716	3.4	1.27489	3.5
	4	0.53532	14.8	0.53384	13.9	0.54374	15.6	0.53351	14.0
	0.25	1.76941	1.4	The above results which do not change for $\lambda = 80$ and 120 coincide practically with those of standard elastica		1.81267	1.2	1.76880	1.4
80	1	1.29976	2.9			1.31802	2.8	1.29928	2.9
	4	0.53542	14.7			0.54396	15.5	0.53376	14.0
	0.25	1.86281	1.0			1.89480	0.9	1.86251	1.0
120	1	1.30478	2.8			1.32243	2.7	1.30446	2.7
	4	0.53543	14.5			0.54400	15.5	0.53380	13.9
		Unsuitable for practical applications		Inconvenient for practical applications		Very convenient for practical applications		Inconvenient for practical applications	

Table 3. Critical loads for $e = -0.0025$ and various values of ρ, μ and λ

λ	μ	ρ	Exact elastica analysis, eqns (10) and (11) β^2_{cr}	Bifurcational stability analysis, eqns (18b) and (28) of Kounadis (1985) β^2_{cr}	Simplified non-linear stability analysis, eqns (18) and (21) β^2_{cr}	Non-linear kinematic stability analysis, eqns (45) and (46) β^2_{cr}
		0.25	1.06210	1.30880	1.11039	1.06120
	0.25	1	0.52901	0.53384	0.53970	0.52872
		4	0.15717	0.15671	0.16001	0.15668
		0.25	1.50954	2.00187	1.57970	1.50766
40	1	1	1.27626	1.30880	1.29716	1.27489
		4	0.53532	0.53380	0.54374	0.53351
		0.25	1.68042	2.30446	1.75747	1.67808
	4	1	1.91165	2.00187	1.93510	1.90879
		4	1.31071	1.30880	1.32404	1.30634
		0.25	1.20716		1.23850	1.20688
	0.25	1	0.53265		0.54291	0.53251
		4	0.15718		0.16003	0.15670
		0.25	1.76941		1.81267	1.76880
80	1	1	1.29976		1.31802	1.29928
		4	0.53542		0.54396	0.53376
		0.25	1.98795	The above results which do not change for $\lambda = 80$ and 120 coincide practically with those of standard elastica	2.03453	1.98718
	4	1	1.97214		1.98928	1.97130
		4	1.31161		1.32561	1.30818
		0.25	1.25403		1.27914	1.25390
	0.25	1	0.53336		0.54352	0.53325
		4	0.15718		0.16003	0.15671
		0.25	1.86281		1.89480	1.86251
120	1	1	1.30478		1.32244	1.30446
		4	0.53543		0.54400	0.53380
		0.25	2.10216		2.13555	2.10178
	4	1	1.98761		2.00303	1.98716
		4	1.31178		1.32591	1.30853

corresponding error of the non-linear kinematic analysis amounts to 0.2%. However, the values of these parameters (implying $A_2 = 16A_1$) correspond to a frame rather with built-up members than members with solid cross-sections. These slight discrepancies between elastica analysis and the last linearized analyses decrease substantially for $\lambda = 80$ becoming 2.3 and 0.03%, respectively. As the slenderness ratio increases further these differences tend to disappear. Contrary to the accuracy of these two stability analyses, the bifurcational stability analysis—based on eqn (30)—which neglects the effect of compressibility of column axis leads to serious errors. The error in the extreme case $\rho = 0.25$ and $\lambda = 40$ amounts to 38.6% which for $\rho = 1$ and $\lambda = 40$ reduces to 7.2%. The results of the last bifurcational analysis are almost identical with those of the standard elastic analysis (Christodolou and Kounadis, 1986). Moreover, it should be noted that the critical displacements of both linearized analyses are quite accurate compared to the corresponding displacements of the exact elastica analysis, being approximately 10,000 greater than those of the bifurcational stability analysis (eqns (18b) and (28) in Kounadis (1985)) and of the above standard elastica. Note that many of the results obtained by the exact elastica analysis have been also derived by using a very efficient, reliable and practical to use approximate stability analysis developed by Kounadis (1986).

In Table 2, one can see the variation of the numerical results of Table 1, in the case of an imperfect frame due to loading eccentricity $e = -0.0025$. For such a case, from the onset of loading the primary equilibrium path is associated with bending. The observations made for the perfect frame (Table 1) are qualitatively the same for the imperfect frame (Table 2); the discrepancies between the exact elastica analysis and the foregoing three linearized analyses are analogous to those of Table 1.

Quite similar observations can be made on the basis of the results presented in Table 3, established for $e = -0.0025$ and various combinations of values of the slenderness ratios λ, length ratios ρ and stiffness ratios μ. From Table 3 one can also see that the slight discrepancies between the two linearized analyses and the exact elastic analysis increase for

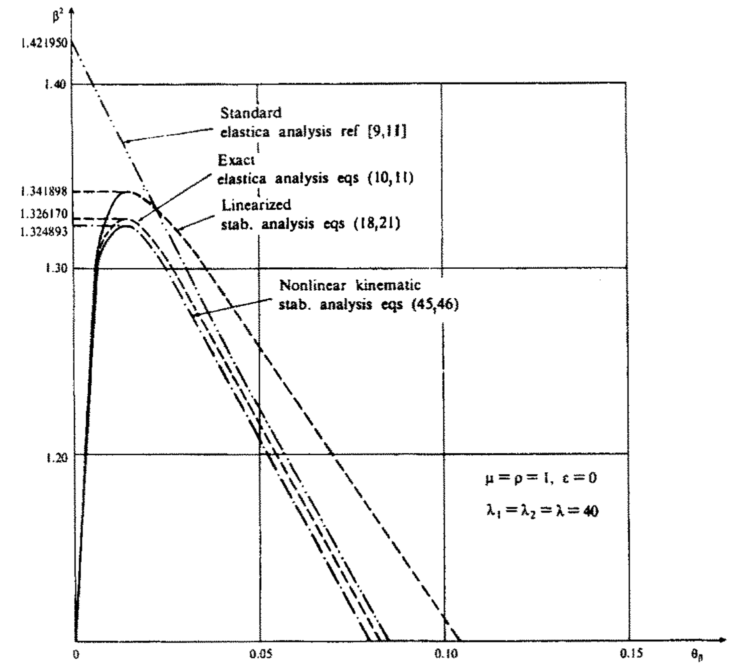

Fig. 2. Loading β^2 vs joint rotation ϑ_B for a frame with $\rho = \mu = 1$, $\lambda_1 = \lambda_2 = \lambda = 40$ and $e = 0$.

low values of the slenderness ratio λ and length ratio ρ, and large values of the stiffness ratio μ. In the worst case for $\lambda_1 = \lambda_2 = 40$ and μ/ρ^2 $(=A_2/A_1) = 64$ the error in the critical loads β_{cr}^2 obtained by the completely linearized stability analysis is much less than 5%.

This analysis based on a simplified mathematical formulation with appreciably reduced nonlinearity of the governing equations, is the simplest possible (analytic) approach for evaluating the initial postbuckling response of frames losing their elastic stability either through a bifurcation or a limit point. This has been also confirmed by comparison with the results of other postbuckling analyses of multistory frames (Economou, 1984; Economou and Kounadis, 1987).

From the "exact" elastic analysis is drawn the important conclusion that the effect of compressibility of the bar axis may have an appreciable influence on the non-linear response of frames having members with low slenderness ratio. A comparison of the above stability analyses can be clearly seen with the aid of the plot β^2 vs joint rotation shown on Fig. 2 corresponding to a frame with $\lambda_1 = \lambda_2 = 40$, $\mu = \rho = 1$ and $e = 0$.

The reliability and range of applicability of any postbuckling analysis or lower order analysis could be checked by comparison with the results presented herein and particularly those of the elastica analysis which includes the effect of axial deformation.

7. CONCLUSIONS

From the above theoretical development and the numerical results based on a simple rectangular frame as well as on those of several multistory frames (analysed successfully by the proposed simplified approach), one may list the following.

(1) For the first time an exact non-linear stability analysis of large elastica response is obtained including the effect of compressibility of the bar axis which may play a significant role on the non-linear response of frames associated with limit point instability.

(2) Using the foregoing exact elastica analysis, the efficiency and range of applicability of three linearized stability analyses are thoroughly discussed. Among the three stability analyses one is completely linear (corresponding to a linear boundary-value problem) and extremely easy to use, the other two are time consuming and impractical for frames with a large number of joints; the less accurate of these analyses is that which neglects the effect of compressibility of the bar axis, while the more accurate is the non-linear kinematic stability analysis yielding almost coincident results with those of the exact elastica analysis.

(3) The completely linearized stability analysis always furnishes results of satisfactory accuracy, being fairly near to those of the widely used non-linear kinematic stability analysis for rectangular frames having members with slenderness ratios greater than 40. The last analysis is valid for moderately large rotations, while the former simplified non-linear analysis holds for somewhat smaller rotations with magnitudes greater or fairly near to those of critical joint rotations of conventional frames. Both these analyses take into account the compressibility effect of the bar axis, whereas the bifurcational stability analysis which neglects this effect (standard elastica) may yield completely inaccurate results for frames having members with low slenderness ratios, or imperfection sensitive frames; this disadvantage along with the computational difficulties render the last analysis unsuitable for use.

(4) The completely linearized stability analysis has the simplest possible formulation (with reduced nonlinearity of the governing equations) and it is easy to use for establishing a reliable evaluation of the initial postbuckling response of frames with many joints; a further linearization would lead to a second-order analysis. Hence, this simplified stability analysis is, for structural design purposes, the most powerful post-buckling analytic approach for framed structures.

(5) The critical limit point loads of the completely linearized stability analysis and the non-linear kinematic stability analysis are always much smaller than those of the bifurcational stability analysis (the standard elastica) which neglects the effect of compressibility of bar axes. The standard elastica can be applied only to rectangular frames having very slender members with $\lambda > 120$ which exhibit in an asymptotic sense a bifurcational instability.

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